Ergodic theory of one-dimensional dynamics

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During the past fifty years a clearer understanding of one-dimensional dynamics has emerged. This paper summarizes the main results of the probabilistic theory of one-dimensional dynamics and shows the behavior to be surprisingly rich and a good starting point for the general theory of dynamics.

For the first time in the history of physics we therefore have a framework with the capacity to explain every fundamental feature upon which the universe is constructed. [. . .] These grandiose descriptive terms are meant to signify the deepest possible theory of physics—a theory which underlies all others, one that does not require or even allow for deeper explanatory base. [. . .] If you understand everything about the ingredients, the reductionist argues, you understand everything. [. . .] Many find it fatuous and downright repugnant to claim that the wonders of life and the universe are mere reflections of microscopic particles engaged in the pointless dance fully choreographed by the law of physics. [. . .] Understanding the behavior of an electron or a quark is one thing; using this knowledge to understand the behavior of a tornado is quite another. [. . .] In fact, the mathematics of (the) [. . .] theory is so complicated that, to date, no one knows the exact equations of the theory. Instead the physicists know only the approximations to these equations, and even the approximate equations are so complicated that they as yet have been only partially solved.
Brian Greene, *The Elegant Universe*, 1999

1. Introduction: What is dynamics?

Simple rules imply a simple world

The observations, experiments, and theoretical work which lay at the basis of contemporary science led to the concept that there are a few fundamental principles which explain the whole of our world. These fundamental principles are expected to be very simple. Until the end of the nineteenth century, it was also believed that these simple rules allow only relatively simple phenomena. All disturbing discrepancies were blamed on measurement error or attributed to some missing factor in the model.

Since the time of Galileo and especially Newton, the way to deal with the description of the physical world is to project it into the imaginary world of mathematics with its (as was long believed) impeccable logic. In particular, the physical world is described by a differential equation with some parameters. After some work, i.e., solving the equation, the conclusions are applied to the physical world. In this way Newton's law of gravitation can explain Galileo's Pisa tower experiments, conclude Copernicus' theory or derive Kepler's laws. Ultimately, this paradigm leads to engineering principles and practice.

The assertion "after some work" hides a tremendous problem. Of course, solving nonlinear differential equations can almost never be done explicitly. The solutions of the models can be fully understood in only very rare cases. Even in the case of a classical solar system with only two planets orbiting a sun, the "after some work" state cannot be achieved.

Sensitivity complicates the world

Unfortunately, the situation is considerably more complicated. Even if the principles are simple, their action can lead to complexities which confuse our understanding and defy any attempt at control. The reason for the breakdown of the simple rules–calculations–understanding/control program is captured by two notions of sensitivity.

The first type, sensitivity to initial conditions, has very practical and dramatic consequences. To introduce this notion, let us discuss how one can find solutions of differential equations.

Because of the lack of analytical tools, the help of numerical analysis is needed. If the systems evolve according to some cyclic (also called periodic) process, the computer will be able to predict this cyclic behavior and compute all of its quantitative aspects.

It was long believed that the solar system, and, indeed, everything can be understood in terms of cyclic behavior. Only at the end of the nineteenth century was it first

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observed by Poincaré that systems can behave in a noncyclic manner. In particular, it was observed that the actual behavior of a system may depend very strongly on the initial data. A small measurement error in the initial data implies very different behavior after some time. This phenomenon of sensitivity to initial conditions is often denoted by the term *chaos*.

Chaos, or sensitivity, forms an essential obstacle. The computer cannot really provide a solution, because the smallest error in initial data usually causes errors that grow exponentially as time elapses. This phenomenon of sensitivity to initial conditions implies that the computer cannot assist us to understand the differential equation by just computing approximations of solutions. Are we really interested in the explicit solutions of such a sensitive system? Instead of looking for explicit solutions, a theory has to be built that predicts the crucial and stable (computable) aspects of the system. Once some understanding of the behavior of the system is attained, that is, once the crucial and stable parts of the system have been identified, numerical analysis can be used to quantify these parts.

The second type of sensitivity a model can have is sensitivity to the parameters of the model. Indeed, there are many examples known in which the slightest measurement error in the parameters causes the model to behave very differently. Clearly, in such a situation we cannot deduce any conclusion from the model for the physical world. Therefore, for any model we use to describe the physical world, we must be able to describe the behavior of nearby models, and we must have the means to deal with the model's sensitivity to initial conditions.

One can think of the study of real-world evolution processes as having two branches. On one hand, there is a wealth of research to define the fundamental principles for the physical world. On the other hand, there is much work aimed at explaining our diverse, complicated, and changing environment. Dynamics is the mathematical study of evolution processes, also called dynamical systems. Its role is to explain and describe this rich behavior.

2. The Palis conjecture

In the study of dynamical systems, the main questions can be captured by one: What is the typical behavior of a typical dynamical system? Notice that this question relates strongly to the difficulties arising from the two types of sensitivity; since it is impossible to understand all possibilities, one must restrict oneself to typical behavior. To answer such a question, one must define a typical system and the elements of "typical behavior"; most significantly, one must classify the phenomena it is possible to observe. This section presents the precise formulation of this question: the Palis conjecture. The goal is to understand chaos. To do this, a vocabulary and tools must be developed to describe and predict whatever is predictable for chaotic systems. This program in general is far from being finished.

Precise information versus a comprehensive overview

The difficulties in the analysis of dynamical systems are of many different natures. We may not be able to solve the equations, as is usually the case for nonlinear differential equations. Even if we solve them, we may have problems in actually calculating the values of the solutions with good accuracy. Even if we can, there may be many different cases, depending on the starting point, and we will be overwhelmed by the amount of specific information and lose control of the overall situation. Problems of this type are intrinsically related to the sensitivity of the dynamical system to initial conditions. When the system describes multiple interacting objects, it is often modeled in a multi-dimensional space where we have poor intuition. Finally, we cannot be sure that our system is precisely the right model and not (in the best case) merely a close approximation. This relates to sensitivity to the parameters of the model.

We now begin to formulate the basic notions used to describe dynamical systems. As examples of dynamical systems, we use differential equations such as Newton's law for the solar system or the Navier–Stokes equation for fluid dynamics, describing evolution in continuous time. There is no fundamental difference between continuous time and discrete time from the point of view of dynamical systems.

A dynamical system

From now on, we think of a dynamical system as a map,

$$f : X \to X.$$

The space X is the state space, and the map f describes how states change from second to second. For example, if the state at time $t = 0$ is x_0, then after n seconds the state is

$$x_n = f(x_{n-1}) = f \circ f(x_{n-2}) = f^2(x_{n-2}) = \cdots = f^n(x_0).$$

The sequence

$$x_0, x_1, x_2, \cdots,$$

starting at the initial state x_0, is called the *orbit* of x_0.

The (state) space has usually some natural additional structures such as a notion of distance between states; that is, the state space is a geometrical object. Often there is a natural notion of probability for a state to occur, which means that the state space is also equipped with a measure (or volume). In typical cases, f is a smooth map on a smooth manifold which carries a natural measure. In

the case of the dynamics of the solar system, a state of it is described by the position and velocities of the different planets and the sun. Each state is a long vector of numbers. The state space is a high-dimensional Euclidean space, and the usual notion of volume is preserved by the dynamics. In the case of fluid dynamics, the state space is a space of functions (vector fields) on some (Euclidean) domain, an infinite-dimensional state space.

Given a dynamical system, it is rather easy to predict the near future; one has to apply the map f five times to predict what happens after five seconds. In principle, this can be done with an arbitrarily high precision. When one deals with a system that is highly sensitive to initial conditions (think of turbulent flows or the weather), it becomes very hard to calculate and predict the longer-term future. The surprising fact is that very long-term behavior can be understood much better. Moreover, in many practical systems the evolution that starts at any typical initial state will show this asymptotic behavior after a rather short period of time. This leads to the notion of an attractor.

Attractors

Definition 1

Let $f : X \to X$ be a dynamical system and let λ be a measure on X. A compact set $A \subset X$ is called an attractor of f if

- It is invariant, $f(A) = A$.
- Its basin of attraction, $B(A)$,

  $$B(A) = \{x \in X | \lim_{n\to\infty} \text{dist}[f^n(x), A] = 0\}$$

  has positive measure $\lambda[B(A)] > 0$.
- There is no smaller set with this property.

The points from the basin of an attractor converge toward the attractor. Often, the basin of attraction is a rather complicated set; it is usually not a simple neighborhood of the attractor. The only criterion for the basin is that there must be a definite probability of picking a state in the basin; it has to have positive measure/volume/area/probability.

The dynamics on the attractor, $f : A \to A$, describes the asymptotic dynamics of the points in its basin. Usually the convergence toward the attractor is exponentially fast. One thinks of an attractor as the site at which the essential aspects of the system appear.

One should look at the attractor(s) of a system when one wants to understand the dynamics of that system. We work on the asymptotics of the system in order to skip any transition artifacts and concentrate on the final pattern. This is a reflection of the belief that the destination is the goal of the journey. But there is more to it. In the dynamics of the falling avalanche, the initial conditions and the final layout are what matters, while the history of the fall is just a means of specifying the outcome. On the other hand, not only the disaster following a tornado but also the shape of the tornado may be seen as an asymptote of the dynamics of the air and water particles. While it is virtually impossible to predict the history of any individual particle, the shape of a tornado seems to evolve stably on a large time scale.

Before introducing the notions needed to describe the behavior in an attractor, we discuss a few basic examples. A simple case of a system is a map $x \mapsto x/2$ on the real line. The point $x = 0$ is fixed, and all other points travel to it geometrically. In this case, we say that the point $x = 0$ is a hyperbolic attractor. When we swap 0 and ∞ by putting $y = 1/x$, we get the system $y \mapsto 2y$, with $y = 0$ a hyperbolic repellor, a fixed point with the neighbors escaping at a geometric rate. One can see this better by looking at the stereographic projection on a circle with the dynamics flowing from one pole to the other. The south pole is the attractor of the system, which attracts all points but the north pole.

Linear and hyperbolic maps

The linear system $x \mapsto x/2$, $y \mapsto 2y$ on the real plane leaves the axes invariant, but with the dynamics described above and outside the axes themselves, there are invariant hyperbolas $xy =$ constant with the trajectories moving from one infinity to the other. The fixed point (0, 0) is called a saddle. A generalization of this example is a hyperbolic linear map—with all eigenvalues nonzero and not on the unit circle—such that in the direction of eigenvectors there is a geometric rate of expansion or contraction.

The following example plays, in spite of its simplicity, a crucial role in the general theory of dynamical systems. An Anosov map is given by

$$A : (x, y) \mapsto (2x + y, x + y) \mod 1,$$

with invariant axes parallel to the eigenvectors of the matrix. This linear system acting on the torus exhibits much more complicated behavior because the invariant axes wrap around densely; moreover, there are many periodic points. Nevertheless, near (0, 0) the system can be modeled as the saddle by the corresponding linear map on the plane; see **Figure 1**.

The area of a domain in the torus plays the role of volume. It is usually called the *Lebesgue measure*. It can be shown that the area of the set of points with orbits which are densely spread throughout the torus is 1. This means that if one picks an initial point at random, with probability 1 it will have a dense orbit. In what follows, we refer to such a crucial phenomenon in the following

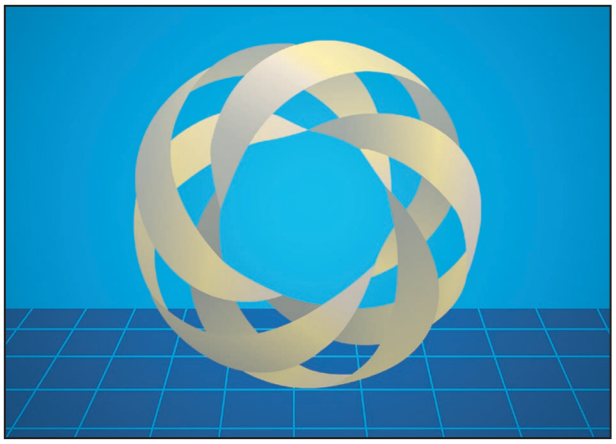

Figure 1

Toral automorphism: Artist's rendition of the figure of an Anosov map acting on a torus which was provided by Roy L. Adler, Matthew Grayson, and Bruce Kitchens, former Research Staff Members of the Mathematical Sciences Department. The bottom half and top half of the torus are distinguished by shading; then the map of the torus is applied twice, and the images of the two regions are followed.

manner: *Almost every point, in the sense of the Lebesgue measure, has a dense orbit.* Notice that the above set of points with a dense orbit has a rather complicated structure because the periodic points are in the complement but are spread around the torus densely.

In other words, the attractor of the Anosov map is the whole torus. In particular, the basin of the attractor is a full measure set in the torus. The relevance of this example is that in some more realistic models the dynamics on the attractor can be understood as Anosov maps.

The linear models above, in Euclidean space or on tori, are very well understood, and as such are good role models. However, there is no way that linear maps can be used to explain general dynamics even by approximation. We just keep them in mind as first simple examples that already show some complications, such as the Anosov maps.

Typical points

Let us continue to build our toolbox. The notion of a typical initial point demands the criterion, "typical in what respect?" In dynamics there are two different kinds of typical initial points. The first type has already been discussed—the notion of *typical with respect to a measure/volume/probability*. A set of initial points consists of typical points if one can pick such an initial point with probability 1 or, equivalently, when the set has full volume/measure.

The second kind of typical initial points, the so-called *generic* points, does not have a probabilistic flavor. It is a topological notion, purely related to space. A set consisting of generic points is "fat" in the sense of densely spread squares in the plane. Notice that it is possible to construct a set of generic points with arbitrarily small area. *A priori* measure theory and topology can show quite different typical phenomena.

An attractor describes the behavior of a typical point from the measure-theoretical point of view [i.e., the basin $B(A)$ is supposed to have positive measure]. One could as well consider the notion of an attractor in which the basin is supposed to be a large topological set.

Of course, we expect that the topological point of view and the measure-theoretical one will both be natural for the system. In particular, we expect that typical behavior from the topological point of view will coincide with typical behavior from the measure-theoretical point of view. People become upset (or excited) when the two notions do not conform, i.e., when there is a large topological set with one behavior and another set with large measure with a different behavior, and they are intermingled. We address this question in more detail during the discussion of one-dimensional dynamics.

An attractor is the place in which the asymptotic behavior of measure-theoretical typical points takes place. The way to describe the behavior of the attractor is of a statistical nature; it is described in terms of time averages of observables. An observable is a function,

$$\phi : X \to \mathbb{R},$$

with some properties related to the structure on the space X (smooth, continuous, integrable, measurable). In the case of some turbulent flow, one can think of the observable $\phi(x)$ to be the temperature or the pressure at a certain spot in the fluid when it is in state x.

Definition 2

A physical measure on the attractor A is a measure μ on A such that for every observable $\phi : X \to \mathbb{R}$ we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi[f^i(x)] = \int \phi \, d\mu,$$

for almost every $x \in B(A)$ with respect to the measure λ on X.

A physical measure is a volume form defined on the attractor. Its crucial property is that it allows one to compute time averages of observables (the left side of the formula above) as space averages of the same observables (the right side of the formula). In particular, the existence of a physical measure states that almost every orbit behaves the same way in a statistical sense.

If one thinks about sensitivity to initial conditions as it exists in chaotic systems, it is very hard to believe that

something like a physical measure can exist. It is also usually very hard to prove. However, in realistic models and in the physical world, such measures are actually observed. One could believe that all macroscopic observations are determined by an underlying physical measure. In the more precise discussion of one-dimensional dynamics, we address the existence of physical measures and related difficulties.

Stochastic stability

Finally, we discuss the fundamental notion of the stability of the system. A model whose features change a great deal under even small perturbations of the model cannot be a realistic model. A realistic model is one in which small perturbations do not change the overall properties of the system. In particular, one needs systems $x_n = f(x_{n-1})$ such that the perturbed system

$$x_{n+1} = f(x_n),$$

$$y_{n+1} = f(y_n) + \delta_n,$$

where the δ_n are small random perturbations of the orbit, behaves essentially like the original system described by f.

Definition 3

Let $A \subset X$ be an attractor of f with a physical measure μ. The system f is stochastically stable on A if the time averages of the perturbed model are close to the space average of the unperturbed model. For every $\epsilon > 0$ there is a $\delta > 0$ such that for all small enough random perturbations [$\|(\delta_n)\| < \delta$, with an appropriate norm],

$$\lim_{n\to\infty} \left| \frac{1}{n} \sum_{i=0}^{n-1} \phi(y_i) - \int \phi \, d\mu \right| < \epsilon,$$

for almost all $y_0 \in B(A)$.

These notions do not require strict evaluations of the orbits or solutions. Generally, the statements have an asymptotic nature; hence, we may say that dynamics is the investigation of typical asymptotic behavior. We do not have to be specific when we predict the future, but we are expected to make an educated guess.

The Palis conjecture

During the last forty years many deep but partial results have been obtained. An important part of this development is the insight that a purely topological study of chaos cannot be adequate. Small topological perturbations of a system too often do not yield to a topologically equivalent system. The measure-theoretical approach forms a crucial aspect of dynamics. This insight was developed from many examples, although a complete understanding is still far from being achieved. The conjecture below summarizes and formalizes for the first time the consensus in the dynamics community.

In the space of smooth dynamical systems on a smooth manifold X as state space there exists a dense set of systems with the following properties:

- There are finitely many attractors whose union of basins is of full measure.
- Each attractor carries a physical measure.
- For typical parameter values, perturbations of such a system by generic families will lead for almost every member of the perturbing family to a perturbed dynamical system which has finitely many attractors whose union of basins is nearly equal to the basin of the initial attractors. These perturbed attractors support a physical measure.
- Each attractor is stochastically stable.

3. One-dimensional dynamics

The attractor is smaller than the space

The structure of the state space has a strong influence on the possible dynamical phenomena one can expect. For example, the Anosov map seen as a map on the plane shows rather trivial behavior. As a map on the torus, it is very complicated. What happens is that the structure of the space limits the global consequences of expansion or of sensitivity to initial conditions. Generally speaking, the higher the dimension of the state space, the more difficult it is to describe the possible dynamical phenomena.

In realistic models, the state space is usually of very high dimension. It might even be of infinite dimension, as in the case of fluid dynamics, in which the states are vector fields. However, realistic models are dissipative because there is friction which causes energy loss. It has been observed, and in many cases proven, that the attractors of these models are actually of lower dimension, much smaller than the dimension of the state space.

In this section we describe one-dimensional dynamics and in particular show that the Palis conjecture holds. In view of the above discussion, one-dimensional dynamics should be the simplest possible; the state space is just an interval with the simplest topology one can imagine. However, there are three surprising reasons for the importance of the study of one-dimensional dynamics.

The first reason is that the observed dynamics of one-dimensional systems is very rich. Most dynamical phenomena observed until now in higher-dimensional dynamics have their counterpart in one-dimensional dynamics. Second, the extensive technology developed to study one-dimensional dynamics turns out to be very useful for the study of higher-dimensional systems. And most surprisingly, predictions from the one-dimensional theory turn out to hold for realistic models of very

high dimension. There is a rather precisely understood phenomenon, the so-called homoclinic bifurcations, which is observed in many realistic models and which explains why one-dimensional phenomena can appear in higher-dimensional dissipative systems.

By no means can one expect one-dimensional dynamics to explain everything. However, the richness of its dynamics and the characteristics it displays in realistic systems make it much more than a simple toy.

Contraction and expansion

Before describing one-dimensional dynamics in more detail, we discuss some examples. A reasonable map from an interval to the same interval either does or does not fold the interval; the graph of a map on the interval is either monotone or not. An interval map which is monotone (i.e., does not fold) has trivial dynamics. One easily sees that every point converges to a fixed point or a periodic point of period 2. Interesting interval dynamics is about folding.

All of the maps we consider below have interesting dynamics in the interval $[-1, 1]$, such that

$$f : [-1, 1] \to [-1, 1].$$

Let us first consider the map defined by

$$f : x \mapsto -\frac{1}{2}|x| - \frac{1}{2}.$$

The graph of this map looks like a low tent. For any $x \in [-1, 1]$, one sees immediately that $\lim_{n\to\infty} f^n(x) = -1$. The attractor of the system is the fixed point -1. A more interesting example is the map defined by

$$f : x \mapsto -2|x| + 1.$$

The graph of this map also resembles a tent, but this one is rather high. The derivative is ± 2; moreover, each small interval has two pre-images that are twice as short. By iterating forward, we immediately notice the sensitivity to initial conditions, and we can even quantify it. The distance between the iterates of two points grows exponentially fast with rate 2,

$$|f^n(x) - f^n(y)| \geq 2^n|x - y|,$$

as long as $f^i(x)$ and $f^i(y)$ are on the same side of 0. Once $f^n(x)$ and $f^n(y)$ are separated by 0, the folding aspect of the map may bring the points back together: The topology places limitations on the consequences of sensitivity to initial conditions. The expansion rate $\log |2| > 0$ is called the Lyapunov exponent of the system.

Probability and the Perron–Frobenius operator

The second example preceding, i.e., the example which is sensitive to initial conditions, has a rather rich statistical behavior. If we consider the left and the right halves of the interval, we may code each point by a sequence of two symbols, L(eft) and R(ight), depending on where the nth iterate of the point lands. This recalls coin tossing, and indeed, when we think of the (normalized) Lebesgue measure as the probability, we see that our system represents a sequence of independent random variables in very much the same way as coin tossing. By probability limit theorems, almost every point will behave asymptotically in the same way whenever the criterion is formulated as an average along the trajectory. For instance, the frequency of visit of a trajectory of almost every point in a set tends to the measure of this set. The whole interval is the attractor, and the Lebesgue measure is the physical measure on the attractor.

More can be said on the statistical behavior of this system. Instead of considering simple states (i.e., points in the interval), one considers a distribution on the state space. That means that each state has a certain probability of being realized. For example, one could use the uniform distribution defined by the Lebesgue measure on the interval. But there are many other distributions one could use, each defined by a density

$$\rho : [-1, 1] \to \mathbb{R},$$

meaning that the probability $p([a, a + \epsilon])$ of picking a state in a small interval $[a, a + \epsilon]$ is

$$p([a, a + \epsilon]) = \int_a^{a+\epsilon} \rho dx \sim \rho(a)\epsilon.$$

Given such a distribution, that is, a (nonnegative) function on the interval, one time step will change this distribution. For example, in the first case above, after each step, the states far away from the fixed point -1 will become less and less likely to be realized. The Perron–Frobenius operator

$$P_f : \{\text{densities}\} \to \{\text{densities}\}$$

describes how the distributions are pushed forward in time by the dynamics of f.

In this context, a physical measure for a system is exactly the measure which is fixed for this operator and attracts all reasonable distributions, like the Lebesgue measure. In particular, the second example above shows the exponential convergence of any density to the Lebesgue measure. This means that the limiting statistical behavior is established rather quickly. This phenomenon clearly relates to the spectral properties of the Perron–Frobenius operator.

Standard unimodal family

As standard examples of one-dimensional systems, we consider the following family of so-called unimodal maps $f_t : [-1, 1] \to [-1, 1]$ defined by

$$f_t(x) = -(1 + t)x^2 + t,$$

with the parameter $t \in [-1, 1]$. One can show that f_1, the map which folds the interval and covers the whole interval, behaves similarly to the high-tent map described above, and is analogous to coin tossing. The corresponding physical measure will not be the Lebesgue measure but one given by the density $1/(\pi\sqrt{1 - x^2})$.

For small parameters t, one again observes that the fixed point -1 attracts every point.

The next section addresses behavior for general parameter values t. This theory applies to all generic similar families. The explicit formulae for the maps are not crucial.

4. The ergodic theory of unimodal maps

It took the last forty years, and contributions of many distinguished mathematicians, to arrive at a clear understanding of one-dimensional dynamics. This development played its role in the formulation of the Palis conjecture.

In this section we describe in more detail the behavior of one-dimensional dynamics. In order, we address the structure of attractors, the existence of physical measures, and finally stability.

Theorem 1

Any standard unimodal map has a unique attractor whose basin is of full measure. There are three possible types: periodic orbit, Cantor set, and cycle of intervals.

The periodic attractors

This is the easy case. There is a point which returns to itself after a while, and the derivative at this point is smaller than (or equal to) 1. This finite orbit is the attractor. The structure of this orbit is very simple, and moreover the structure of its basin is very simple. Around each point in this periodic orbit there is a small interval contained in the basin, and these form the so-called immediate basin, which is a cycle of intervals invariant by the dynamics. The preimages of the immediate basin form an open and dense set of full measure. However, there is usually a compact set of measure zero which is not in the basin. The dynamics in this set is usually chaotic but it is not visible; these chaotic points are not typical.

The unimodal maps with a periodic attractor are called *structurally stable*, which means that the system can be perturbed slightly with no essential change in its behavior. In particular, the hyperbolic periodic attractor (with the derivative less than 1) persists. This topological form of stability is much stronger than the stochastic stability discussed before.

A final remark on this case is that the period of the periodic attractor can be arbitrarily long. In such cases, very careful experiments are needed to identify whether or not there is a periodic attractor. The attractor might become fairly densely spread, and the experiment might indicate attractors of more complicated structure.

The Cantor attractor

A Cantor set in the interval is a nonempty compact set which has no isolated points and which does not contain intervals. In other words, each point in a Cantor set can be arbitrarily close to other points in the Cantor set, but two points can never be connected by an arc contained in the Cantor set. The standard example of such a set is the well-known middle third Cantor set. The second type of attractor a unimodal map can have is a Cantor set. One can think of such an attractor as a limiting case when one considers periodic attractors for which the period grows to infinity.

As we saw before, a periodic attractor might be spread fairly densely through the interval. This means that we have to be more careful when describing a Cantor attractor as a limiting case of periodic attractors. The situation is rather beautiful. To understand the structure of such a Cantor attractor, consider the immediate basin of a periodic attractor with a very long period. It turns out that one can replace the periodic attractor with a periodic attractor of twice the length; the new attractor "shadows" the original attractor twice. The basin of this newly constructed attractor winds twice around the original basin. Repeat this construction (changing the new periods *ad libitum*), and in the limit one finds a Cantor attractor. The basin of such an attractor is an open and dense set of full measure (as with a periodic attractor). The measure of the attractor itself is zero.

The Cantor attractors of this form play a special role in one-dimensional dynamics. The corresponding maps are called *infinitely renormalizable*. One of their important properties is that the microscopic geometry of such attractors is independent of the actual system under consideration. This geometry is referred to as being universal. One of the surprises of one-dimensional dynamics is that this universal geometry predicted by one-dimensional theory is actually observed in many real-world physical experiments.

Strange attractors

There are *strange Cantor attractors*, which cannot be understood as limits of periodic attractors. Their Lebesgue measure is again zero and their basin, of full measure, does not contain any interval. In particular, many points in the complement of the basin have orbits which are dense. This is a very strange situation. From a measure-

theoretical point of view, one sees that almost every point is attracted toward a Cantor set of Lebesgue measure zero. But from a topological point of view, one sees that a generic point has an orbit which is densely spread throughout the interval. The topology and measure theory give opposite typical behavior. These strange attractors do not exist in the quadratic family we are considering, but they can be found in families such as

$$f_t(x) = -(1 + t)x^a + t,$$

when a is chosen large enough, i.e., when the maximum of f_t is very flat.

Interval attractor

The third type of attractor is a cycle of periodic intervals. This means that there is a small interval which after a while is mapped back into itself in a folded way. The orbit of such an interval is called a cycle. The structure of the basin of attraction is similar to the structure of the basin of attraction of a periodic attractor: It is open and dense and of full measure. The immediate basin of a periodic attractor corresponds to the cycle. However, the dynamics inside the cycle is very different from that inside the immediate basin of a periodic attractor. Almost every point, topologically generic or measure-theoretically typical, will have a dense orbit within the cycle.

Typical unimodal maps and exceptions

Theorem 2
In the quadratic family

$$f_t(x) = -(1 + t)x^2 + t,$$

almost every map (in the Lebesgue sense for the parameter t) has a physical measure. It is one of two types, either concentrated on a periodic attractor or an absolutely continuous measure on an attracting cycle of periodic intervals.

The parameters for which the systems have a periodic attractor with their physical measure form an open and dense set in parameter space. This means that any map can be perturbed into a map with a periodic attractor by arbitrarily small perturbations. The parameters for which the map does not have a periodic attractor form a set which does not contain any interval; it is topologically small but has a positive measure. Hence, picking a parameter at random gives a substantial probability that one will find a system that does not have a periodic attractor. Almost every such parameter will show a system with a physical measure on a cycle of intervals.

In parameter space, the maps with a Cantor attractor form a set of measure zero. However, these maps also have a physical measure on the attractor. On the other hand, there are systems in the quadratic family with one of the following properties: a) the system has no physical measure, or b) the system has a physical measure which is supported on a very small part of the attractor. An example of the second case is a system which has its physical measure sitting on an expanding fixed point. The situation is as follows. Orbits are pushed away from this fixed point, but the map brings orbits back to it rather rapidly, such that most of the time an orbit is close to this expanding fixed point. However, maps with such strange statistical behavior are hard to find (from a measure-theoretical point of view). In these exceptional situations there are still residual sets of dense orbits.

Lyapunov exponents

There is a strong relation between properties of the physical measures and sensitivity to initial conditions. Remember that the Lyapunov exponent λ_f of a system is the rate at which nearby points are separated when time flows. It may be defined by $\lim \Sigma \log |f'(x_i)|/n$ or by $\int \log |f'| \, d\mu$.

In the case of a system with a hyperbolic periodic attractor, the system is not chaotic at all. Actually, two points which are close to the attractor will actually contract toward each other (and the attractor) with a certain rate

$$\lambda_{\text{per}} < 0.$$

On the other hand, a system has an absolutely continuous physical measure, acpm, if and only if

$$\lambda_{\text{acpm}} > 0.$$

This means that systems with such a physical measure are really chaotic, and there is a definite expansion rate. The systems which have no physical measure, or a strange one, are chaotic, but the expansion rate is zero.

A final remark with respect to physical measures is that the case with periodic attractors gives rise to predictable behavior. On the other hand, if the system has a continuous physical measure, the system is chaotic, and from a practical point of view it is not predictable. However, the statistical behavior, i.e., the dynamics of the Perron–Frobenius operator on distributions, is predictable.

Theorem 3
In the quadratic family

$$f_t(x) = -(1 + t)x^2 + t,$$

almost every map, in the Lebesgue sense, is stochastically stable.

The theory of dynamical systems does not pretend to have the capacity to explain every fundamental feature upon which the universe is constructed. However, it has produced tools such as Theorem 3 to guide us through the global picture without exact equations. With respect

to the details, the situation is still far from clear. The one-dimensional theory is a powerful tool in the study of global dynamics. It is rich in phenomena, and although it is complicated, it is quite well understood. As such, it is an extremely solid step toward the understanding of higher-dimensional dynamics.

5. Summary

The study of complicated evolution processes is far from complete, even where such processes are generated by apparently simple models. For example, the movement of the solar system and the flow of a fluid are described by a couple of equations with a few elementary operations, yet there are not enough analytical tools to completely solve these equations. Also, the phenomena which occur in the study of ordinary and partial differential equations (e.g., in Newton's celestial mechanics and Navier–Stokes fluid dynamics) form such a zoo of curiosities that there exists no systematic understanding of all possible phenomena.

One-dimensional dynamics is the study of dynamical systems, or evolution processes, whose state space is one-dimensional. Despite the simplicity of such one-dimensional systems, their behavior is surprisingly rich: Many phenomena known in general dynamical systems have their counterparts in one-dimensional dynamics. Moreover, some universal quantitative properties predicted for one-dimensional systems have been precisely measured in actual physical systems.

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Bibliography and suggested reading

The literature on dynamics and one-dimensional dynamics in particular is extensive. For a general reference, see the following:

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